Derandomizing Isolation Lemma for $K_{3,3}$-free and K_5-free Bipartite Graphs

Rahul Arora[1,4], Ashu Gupta[2], Rohit Gurjar[*3], and Raghunath Tewari[4]

1 **University of Toronto**, `arorar@cs.toronto.edu`
2 **University of Illinois at Urbana-Champaign**, `agupta80@illinois.edu`
3 **Aalen University**, `rgurjar@iitk.ac.in`
4 **Indian Institute of Technology Kanpur**, `{arorar,rtewari}@iitk.ac.in`

Abstract

The perfect matching problem has a randomized NC algorithm, using the celebrated Isolation Lemma of Mulmuley, Vazirani and Vazirani. The Isolation Lemma states that giving a random weight assignment to the edges of a graph ensures that it has a unique minimum weight perfect matching, with a good probability. We derandomize this lemma for $K_{3,3}$-free and K_5-free bipartite graphs. That is, we give a deterministic log-space construction of such a weight assignment for these graphs. Such a construction was known previously for planar bipartite graphs. Our result implies that the perfect matching problem for $K_{3,3}$-free and K_5-free bipartite graphs is in SPL. It also gives an alternate proof for an already known result – reachability for $K_{3,3}$-free and K_5-free graphs is in UL.

1998 ACM Subject Classification F.1.3 Complexity Measures and Classes, G.2.2 Graph Theory

Keywords and phrases bipartite matching, derandomization, isolation lemma, SPL, minor-free graph

1 Introduction

The perfect matching problem is one of the most extensively studied problem in combinatorics, algorithms and complexity. In complexity theory, the problem plays a crucial role in the study of parallelization and derandomization. In a graph $G(V, E)$, a *matching* is a set of disjoint edges and a matching is called *perfect* if it covers all the vertices of the graph. Edmonds [12] gave the first polynomial time algorithm for the matching problem. Since then, there have been improvements in its sequential complexity [24], but an NC (efficient parallel) algorithm for it is not known. The perfect matching problem has various versions:

- Decision-PM: Decide if there exists a perfect matching in the given graph.
- Search-PM: Construct a perfect matching in the given graph, if it exists.

A randomized NC (RNC) algorithm for Decision-PM was given by [23]. Subsequently, Search-PM was also shown to be in RNC [18, 25]. The solution of Mulmuley et al. [25] was based on the powerful idea of *Isolation Lemma*. They defined a notion of an isolating weight assignment on the edges of a graph. Given a weight assignment on the edges, weight of a matching is defined to be the sum of the weights of all the edges in it.

▶ **Definition 1** ([25]). For a graph $G(V, E)$, a weight assignment $\mathbf{w}\colon E \to \mathbb{N}$ is isolating if G either has a unique minimum weight perfect matching according to w or has no perfect matchings.

* Supported by TCS Research Fellowship

The Isolation Lemma states that a random integer weight assignment (polynomially bounded) is isolating with a good probability. Other parts of the algorithm in [25] are deterministic. They showed that if we are given an isolating weight assignment (with polynomially bounded weights) for a graph G, then a perfect matching in G can be constructed in NC^2. Later, Allender et al. [2] showed that the Decision-PM would be in SPL, which is in NC^2, if an isolating weight assignment can be constructed in L (see also [9]). A language L is in the class SPL if its characteristic function $\chi_L \colon \Sigma^* \to \{0,1\}$ can be (log-space) reduced to computing determinant of an integer matrix.

Derandomizing the Isolation Lemma remains a challenging open question. A general version of Isolation Lemma has also been studied, where one has to ensure a unique minimum weight set in a (non-explicitly) given family of sets (or multisets). Arvind and Mukhopadhyay [4] have shown that derandomizing this version of Isolation Lemma would imply circuit size lower bounds. While Reinhardt and Allender [26] have shown that derandomizing Isolation Lemma for some specific families of paths in a graph would imply $\mathsf{NL} = \mathsf{UL}$.

With regard to matchings, Isolation Lemma has been derandomized for some special classes of graphs: planar bipartite graphs [9, 29], constant genus bipartite graphs [10], graphs with small number of matchings [14, 1] and graphs with small number of nice cycles [15]. In a result subsequent to this work, Fenner et al. [13] achieved an almost complete derandomization of the isolation lemma for bipartite graphs. They gave a deterministic construction but with quasi-polynomially large weights. A graph G is bipartite if its vertex set can be partitioned into two parts V_1, V_2 such that any edge is only between a vertex in V_1 and a vertex in V_2. A graph is planar if it can be drawn on a plane without any edge crossings.

It is well known that a graph is planar if and only if it is both $K_{3,3}$-free and K_5-free [33]. For a graph H, G is an H-free graph if H is not a minor of G. $K_{3,3}$ is the complete bipartite graph with $(3,3)$ nodes and K_5 is the complete graph with 5 nodes. A natural generalization of planar bipartite graphs would be $K_{3,3}$-free bipartite graphs or K_5-free bipartite graphs. We make a further step towards the derandomization of Isolation Lemma by derandomizing it for these two graph classes. Note that these graphs are not captured by the classes of graphs mentioned above. In particular, a $K_{3,3}$-free or K_5-free graph can have arbitrarily high genus, exponentially many matchings or exponentially many nice cycles.

▶ **Theorem 2.** *Given a $K_{3,3}$-free or K_5-free bipartite graph, an isolating weight assignment (polynomially bounded) for it can be constructed in log-space.*

Another motivation to study these graphs came from the fact that Count-PM (counting the number of perfect matchings) is in NC^2 for $K_{3,3}$-free graphs [31] and in TC^2 ($\subseteq \mathsf{NC}^3$) for K_5-free graphs [28]. These were the best known results for Decision-PM too. The counting results, together with the known NC-reduction from Search-PM to Count-PM (for bipartite graphs) [20], implied an NC algorithm for Search-PM. Thus, a natural question was to find a direct algorithm for Search-PM via isolation, which we do here. One limitation of the earlier approach is that Count-PM is $\#\mathcal{P}$-hard for general bipartite graphs. Thus, there is no hope of generalizing this approach to work for all graphs. While the isolation approach can potentially lead to a solution for general/bipartite graphs.

Theorem 2 together with the results of Allender et al. [2] and Datta et al. [9] gives us the following results about matching.

▶ **Corollary 3.** *For a $K_{3,3}$-free or K_5-free bipartite graph,*

- Decision-PM *is in* SPL.
- Search-PM *is in* $\mathsf{FL}^{\mathsf{SPL}}$.

– Min-Weight-PM *is in* $\mathsf{FL}^{\mathsf{SPL}}$.

$\mathsf{FL}^{\mathsf{SPL}}$ is the set of function problems which can be solved by a log-space Turing machine with access to an SPL oracle. Like SPL, $\mathsf{FL}^{\mathsf{SPL}}$ also lies in NC^2. The problem Min-Weight-PM asks to construct a minimum weight perfect matching in a given graph with polynomially bounded weights on its edges.

The crucial property of these graphs, which we use, is that their 4-connected components are either planar or small sized. This property has been used to reduce various other problems on $K_{3,3}$-free or K_5-free graphs to their planar version, e.g. graph isomorphism [11], reachability [30]. However, their techniques do not directly work for the matching problem. There has been an extensive study on more general minor-free graphs by Robertson and Seymour. In a long series of works, they gave similar decomposition properties for these graphs [27]. Our approach for matching can possibly be generalized to H-free graphs for a larger/general graph H.

Our techniques: We start with the idea of Datta et al. [9] which showed that a skew-symmetric weight function on the edges ($w(u,v) = -w(v,u)$) such that every cycle has a nonzero circulation (weight in a fixed orientation) implies isolation of a perfect matching in bipartite graphs. To achieve nonzero circulation in a $K_{3,3}$-free or K_5-free graph, we work with its 3-connected or 4-connected component decomposition given by [33, 5], which can be constructed in log-space [30, 28]. The components are either planar or constant-sized and share a pair/triplet of vertices. These components form a *tree structure*, when each component is viewed as a node and there is an edge between two components if they share a pair/triplet. For any cycle C in the graph, we break it into its fragments contained within each of these components, which we call *projections* of C. Any such projection can be made into a cycle by adding virtual edges for separating pairs/triplets in the corresponding component.

Circulation of any cycle can be seen as a sum of circulations of its projections. The projections of a cycle can have circulations with opposite signs and thus, can cancel each other. To avoid this cancellation, we observe that the components, where a cycle has a non-empty projection form a subtree of the component tree. The idea is to assign edge weights using a different scale for each level of nodes in the tree. This ensures that for any subtree, its root node will contribute a weight higher than the total weight from all its other nodes. To avoid any cancellations within a component, weights in a component are given by modifying some known techniques for planar graphs [9, 19] and constant sized graphs.

This idea would work only if the component tree has a small depth, which might not be true in general. Thus, we create an $O(\log n)$-depth *working tree* by finding 'centers' for the component tree and its subtrees recursively. The construction of such a balanced working tree has been studied in context of evaluating arithmetic expressions [7]. In the literature, this construction is also known as 'centroid decomposition' or 'recursive balanced separators'. Its log-space implementation is more involved.

As the working tree has $O(\log n)$ depth, the straightforward way of using a different scale for each level will lead to edge weights being $n^{O(\log n)}$. So instead, in a component node, we assign weights to only those edges which surround a separating pair/triplet. The weighting scheme ensures that the total weight grows only by a constant multiple, when we move one step higher in the working tree.

Achieving non-zero circulation in log-space also puts directed reachability in UL [26, 6, 29]. Thus, we get an alternate proof for the result – directed reachability for $K_{3,3}$-free and K_5-free graphs is in UL [30].

In Section 2, we introduce the concepts of nonzero circulation, clique-sum, graph decomposition and the corresponding component tree. In Section 3, we give a log-space construction of a weight assignment with nonzero circulation for every cycle, for a class of graphs defined via clique-sum operations on planar and constant-sized graphs. This class contains $K_{3,3}$-free and K_5-free graphs (a proof can be found in the full version of this paper [3]).

2 Preliminaries

Let us first define a skew-symmetric weight function on the edges of a graph. For this, we consider the edges of the graph directed in both directions. We call this directed set of edges $\vec{E}$. A weight function $w\colon \vec{E} \to \mathbb{Z}$ is called skew-symmetric if for any edge (u, v), $w(u, v) = -w(v, u)$.

▶ **Definition 4** (Circulation). For a cycle C, whose edges are given by $\{(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k), (v_k, v_1)\}$, its circulation is defined to be $w(v_1, v_2) + w(v_2, v_3) + \cdots + w(v_k, v_1)$.

Clearly, as our weight function is skew-symmetric, changing the orientation of the cycle only changes the sign of the circulation. The following lemma [29, Theorem 6] gives the connection between nonzero circulations and isolation of a matching. For a bipartite (undirected) graph $G(V_1, V_2, E)$, a skew-symmetric weight function $w\colon \vec{E} \to \mathbb{Z}$ on its edges has a natural interpretation on the undirected edges as $\mathbf{w}\colon E \to \mathbb{Z}$ such that $\mathbf{w}(u, v) = w(u, v)$, where $u \in V_1$ and $v \in V_2$.

▶ **Lemma 5** ([29]). *Let $w\colon \vec{E} \to \mathbb{Z}$ be a skew-symmetric weight function on the edges of a bipartite graph G such that every cycle has a non-zero circulation. Then, $\mathbf{w}\colon E \to \mathbb{Z}$ is an isolating weight assignment for G.*

The bipartiteness assumption is needed only in the above lemma. We will construct a skew-symmetric weight function that guarantees nonzero circulation for every cycle, for a given $K_{3,3}$-free or K_5-free graph, i.e. without assuming bipartiteness.

2.1 Clique-sum

First, we will construct a nonzero circulation weight assignment for a special class of graphs, defined via a graph operation called *clique-sum*.

▶ **Definition 6** (Clique-sum). Let G_1 and G_2 be two graphs each containing a clique (of the same size). A clique-sum of graphs G_1 and G_2 is obtained from their disjoint union by identifying pairs of vertices in these two cliques to form a single shared clique, and by possibly deleting some of the edges in the clique. It is called a k-clique-sum if the cliques involved have at most k vertices.

One can form clique-sums of more than two graphs by a repeated application of clique-sum operation on two graphs. Using this, we define a new class of graphs. Let $\mathcal{P}_c$ be the class of all planar graphs together with all graphs of size at most c, where c is a constant. Define $\langle \mathcal{P}_c \rangle_k$ to be the class of graphs constructed by repeatedly taking k-clique-sums, starting from the graphs which belong to the class $\mathcal{P}_c$. In other words, it is the closure of $\mathcal{P}_c$ under k-clique sums. The starting graphs are called the component graphs. We will construct a nonzero circulation weight assignment for the graphs which belong to the class $\langle \mathcal{P}_c \rangle_3$.

Taking 1-clique-sum of two graphs will result in a graph which is not biconnected. For the perfect matching problem, one can assume without loss of generality that the given graph is biconnected (see the full version [3] for details). Thus, we assume that every clique-sum operation involves either 2-cliques or 3-cliques. A 2-clique which is involved in a clique-sum operation is called a separating pair. Similarly, a 3-clique is called a separating triplet. In general, they are called separating sets. Note that deletion of any separating pair/triplet will make the graph disconnected. We emphasize here that there can be other pairs/triplets in the graph which are not involved in a clique-sum operation, but whose deletion will make the graph disconnected. In this work, the term separating pair/triplet does not refer to such pairs/triplets.

2.2 Component Tree

In general, clique-sum operation can be performed many times using the same separating set. In other words, many components can share a separating set. One can modify a graph in class $\langle \mathcal{P}_c \rangle_3$ via some matching preserving operations such that on decomposition, any separating set is shared by only two components (see the full version [3] for details). Henceforth, in this section we assume this property.

Using this assumption, we can define a component graph for any graph $G \in \langle \mathcal{P}_c \rangle_3$ as follows: each component is represented by a node and two such nodes are connected by an edge if the corresponding components share a separating set. Observe that this component graph is actually a tree. This is because when we take repeated clique-sums, a new component can be attached with only one of the already existing components, as a clique will be contained within one component. In literature [16, 30], the component tree also contains a node for each separating set and it is connected by all the components which share this separating set. But, here we can ignore this node as we have only two sharers for each separating set.

In the component tree, each component is shown with all the separating sets it shares with other components. Thus, a copy of a separating set is present in both its sharer components. Moreover, in each component, a separating set is shown with a virtual clique, i.e., a virtual edge for a separating pair and a virtual triangle for a separating triplet. These virtual cliques represent the paths between the nodes via other components If any two vertices in a separating set have a real edge in G, then that real edge is drawn in one of the sharing components, parallel to the virtual edge. Note that while a vertex can have its copy in two components, any real edge is present in exactly one component.

3 Nonzero Circulation

In this section, we construct a nonzero circulation weight assignment for a given graph in the class $\langle \mathcal{P}_c \rangle_3$, provided that the component tree and the planar embeddings of the planar components are given. Moreover, to construct this weight assignment we will make some assumptions about the given graph and its component tree.

1. In any component, a vertex is a part of at most one separating set.
2. Each separating set is shared by at most two components.
3. Any virtual triangle in a planar component is always a face.

Given a $K_{3,3}$-free or K_5-free graph, a component tree can be constructed which has these properties (see the full version [3] for details). The third property comes naturally, as the inside and outside parts of any virtual triangle can be considered as different components sharing this separating triplet. All these constructions are in log-space.

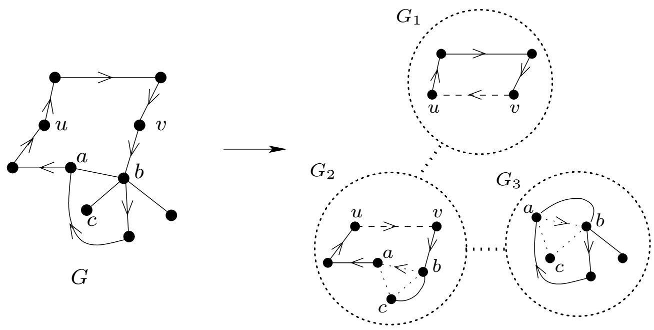

Figure 1 Breaking a cycle into its component cycles (projections) in the component tree. Notice that the original cycle and its components share the same set of *real* edges.

3.1 Components of a cycle

We look at a cycle in the graph as *sum* of many cycles, one from each component the cycle passes through. Intuitively, the original cycle is *broken* at the separating set vertices which were part of the cycle, thereby generating fragments of the cycle in various nodes of the component tree. In all the component nodes containing these fragments, we include the virtual edges of the separating sets in question to complete the fragment into a cycle, thus resulting in component cycles in the component nodes (see Figure 1).

Consider a directed cycle $C = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_0)\}$ in a graph $G = (V, E)$. Without loss of generality, consider that G is separated into two components G_1 and G_2 via a separating pair (v_i, v_0) or a separating triplet (v_i, v_0, u), where $1 \leq i < k$ and $u \in V$. Then, one of the components, say G_1, will contain the vertices $v_i, v_{i+1 \bmod k}, \ldots, v_{k-1}, v_0$, and the other ($G_2$) will contain the vertices $v_0, v_1, \ldots, v_{i-1}, v_i$. Then the cycles $C_1 = \{(v_i, v_{i+1 \bmod k}), \ldots, (v_{k-1}, v_0), (v_0, v_i)\}$ and $C_2 = \{(v_0, v_1), \ldots, (v_{i-1}, v_i), (v_i, v_0)\}$ in G_1 and G_2 respectively are the component cycles of C, and we say that C is the sum of C_1 and C_2. Observe that the edges (v_i, v_0) and (v_0, v_i) are virtual.

Repeat the processes recursively for C_1 and C_2 until no separating set breaks a cycle component, and we get the component cycles of the cycle C. Note that any edge in a cycle C is contained in exactly one of its component cycles. Moreover for any component cycle, all its edges, other than the virtual edges, are contained in C.

Observe that for any separating set in a component, a cycle can use one of its vertices to go out of the component and another vertex to come in (this transition is represented by a virtual edge in the component). As any separating set has size at most 3, a cycle can visit a node of the component tree only once. In other words, a cycle can have only one component cycle in any component tree node (this would not be true if we had separating sets of size 4). Also, a component cycle can take only one edge of any virtual triangle.

▶ **Definition 7** (Projection of a cycle)**.** For a given component node N in the component tree, the component cycle of a cycle C in N is called the projection of C on N. If there is no component cycle of C in N, then C is said to have an empty projection on N.

It is easy to see that for any cycle C, the components on which C has a non-empty projection, form a subtree of the component tree. To construct the weight assignment (Section 3.2), we will work with the component nodes of the component tree. Within any component, weight of a virtual edge will always be set to zero. Along with the fact that each cycle has the same set of real edges as the union of the edges in all its projections, this leads to the following lemma.

▶ **Lemma 8.** *The circulation of a cycle is the sum of circulations of its component cycles.*

Note that for a cycle, its component cycles can have circulations with different signs (positive or negative) as they can have different orientations (clockwise or anti-clockwise) in the planar components. Hence the total circulation can potentially be zero. Our idea is to ensure that one of the component cycles get a circulation greater than all the other component cycles put together. This will imply a nonzero circulation.

3.2 Weighting Scheme

The actual weight function we employ is a combination of two weight functions w_0 and w_1. They are combined with an appropriate scaling so that they do not interfere with each other. w_1 ensures that all the cycles which are within one component have a non-zero circulation and w_0 ensures that all the cycles which project on at least two components have a non-zero circulation. We first describe the construction of w_0.

Working Tree: The given component tree can have arbitrary depth, while our weight construction would need the tree-depth to be $O(\log n)$. Thus, we re-balance the tree to construct a new *working tree*. It is a rooted tree which has the same nodes as the component tree, but the edge relations are different. The working tree, in some sense, 'preserves' the subtree structure of the original tree.

For a tree S, its working tree $\mathsf{wt}(S)$ is constructed as follows: Find a 'center' node $c(S)$ in the tree S and mark it as the root of the working tree, $r(\mathsf{wt}(S))$. Deleting the node $c(S)$ from the tree S would give a set of disjoint trees, say $\{S_1, S_2, \dots, S_k\}$. Apply this procedure recursively on these trees to construct their working trees $\mathsf{wt}(S_1), \mathsf{wt}(S_2), \dots, \mathsf{wt}(S_k)$. Connect each $\mathsf{wt}(S_i)$ to the root $r(\mathsf{wt}(S))$, as a subtree. In other words, $r(\mathsf{wt}(S_i))$ is a child of $r(\mathsf{wt}(S))$. For the base case, when the tree is a node, its working tree is the node itself. This completes the construction. If the component $c(S)$ shares the separating set τ_i with S_i, then the subtree $\mathsf{wt}(S_i)$ is said to be attached to the root $r(\mathsf{wt}(S))$ at τ_i.

The 'center' nodes are chosen in a balanced way so that the working tree depth is $O(\log n)$. Von Braunmühl and Verbeek [32], and later Limaye et al. [21], gave a log-space construction of such a balanced tree, but in terms of well-matched strings (also see [8]). In Section 3.3, we present the log-space construction in terms of a tree, along with a precise definition of a 'center' node.

Note that for any two nodes $v_1 \in S_i$ and $v_2 \in S_j$ such that $i \neq j$, $\mathrm{path}(v_1, v_2)$ in S passes through the node $c(S) = r(\mathsf{wt}(S))$. Thus, we get the following property for the working tree.

▶ Claim 9. For any two nodes $u, v \in S$, let their least common ancestor in the working tree $\mathsf{wt}(S)$ be the node a. Then $\mathrm{path}(u, v)$ in the tree S passes through a.

The root $r(\mathsf{wt}(S))$ of the working tree $\mathsf{wt}(S)$ is said to be at depth 0. For any other node in $\mathsf{wt}(S)$, its depth is defined to be one more than the depth of its parent. Henceforth, depth of a node will always mean its depth in the working tree. From Claim 9, we get the following.

▶ Claim 10. Let S' be an arbitrary subtree of S, with its set of nodes being $\{v_1, v_2, \dots, v_k\}$. There exists $i^* \in \{1, 2, \dots, k\}$ such that for any $j \in [k]$ with $j \neq i^*$, v_j is a descendant of v_{i^*} in the working tree $\mathsf{wt}(S)$.

Proof. Let d^* be the minimum depth of any node in S', and let v_{i^*} be a node in S' with depth d^*. We claim that every other node in S' is a descendant of v_{i^*} in the working tree $\mathsf{wt}(S)$. For the sake of contradiction, let there be a node $v_j \in S'$ which is not a descendant of v_{i^*}. Then, the least common ancestor of v_j and v_{i^*} in $\mathsf{wt}(S)$ must have depth strictly smaller than d^*. By Claim 9, this least common ancestor must be present in the tree S'. But, we assumed d^* is the minimum depth value in S'. Thus, we get a contradiction. ◀

This claim plays a crucial role in our weight assignment construction, as for any cycle C the components with a non-empty projection of C form a subtree of the component tree. To assign weights in the graph, we work with the working tree of its component tree. Let the working tree be $\mathcal{T}$. We start by assigning weight to the nodes having the largest depth, and move up till we reach depth 0, that is, the root node $r(\mathcal{T})$. The idea is that for any cycle C, its unique least-depth projection should get a circulation higher than the total circulation of all its other projections.

Complementary to the depth, we also define *height* of every node in the working tree. Let the maximum depth of any node in the working tree be D. Then, the height of a node is defined to be the difference between its depth and $D + 1$.

Circulations of cycles spanning multiple components: For any subtree T of the working tree $\mathcal{T}$, the weights to the edges inside the component $r(T)$ will be given by two different schemes depending on whether the corresponding graph is planar or constant sized.

Let the maximum possible number of edges in a constant sized component be m. Then, let K be a constant such that $K > \max(2^{m+2}, 7)$. Also, suppose that the height of a node N is given by the function $h(N)$, and the number of leaves in subtree T is given by $l(T)$. Lastly, suppose the set of subtrees attached at $r(T)$ is $\{T_1, T_2, \ldots, T_k\}$.

Constant sized graph: Let the set of (real) edges of the graph be $\{e_1, e_2, \ldots, e_m\}$. The edge e_j will be given weight $2^j \times K^{h(r(T))-1} \times l(T)$ for an arbitrarily fixed direction. The intuition behind this scheme is that powers of 2 ensure that sum of weights for any non-empty subset of edges remain nonzero even when they contribute with different signs.

Planar graph: We work with a given planar embedding of the graph. For any weight assignment $w : \vec{E} \to \mathbb{Z}$ on the edges of the graph, we define the *circulation of a face* as the circulation of the corresponding cycle in the clockwise direction, i.e., traverse the boundary edges of the face in the clockwise direction and take the sum of their weights. Instead of directly assigning edge weights, we will fix circulations for the inner faces of the graph. As we will see later, fixing positive circulations for all inner faces will avoid any cancellations. Lemma 14 describes how to assign weights to the edges of a planar graph to get the desired circulation for each of the inner faces.

Assigning circulations to the faces: Here, only those inner faces are assigned nonzero circulations which are adjacent to some separating pair/triplet shared with a subtree. This is a crucial idea. As we will see in Lemma 11, this ensures that the maximum possible circulation of a cycle grows only by a constant multiple as we move one level higher up in the working tree.

If T is a singleton, i.e., there are no subtrees attached at T, we give a zero circulation to all the faces (and thus zero weight to all the edges) of $r(T)$. Otherwise, consider a separating pair $\{a, b\}$ where a subtree T_i is attached to $r(T)$. The two faces adjacent to the virtual edge (a, b) will be assigned circulation $2 \times K^{h(r(T_i))} \times l(T_i)$. Similarly, consider a triplet $\{a, b, c\}$ where a subtree T_j is attached. Then all the faces (at most 3) adjacent to the virtual triangle $\{a, b, c\}$ get circulation $2 \times K^{h(r(T_j))} \times l(T_j)$. Repeat this procedure for all the faces adjacent to any pairs and/or triplets where subtrees are attached. If a face is adjacent to more than one virtual edge/triangle, then we just take the sum of different circulations due to each virtual edge/triangle.

Recall that by definition, each face has a positive circulation in the clockwise direction. The intuition behind this scheme is the following: circulation of any cycle in the planar component is just the sum of circulations of the faces inside it (Claim 12). As all of them have the same sign, they cannot cancel each other. Moreover, it will be ensured that the contribution to the circulation from this planar component is higher than the total contribution from all its subtrees, and thus, cannot be canceled.

Now, we formally show that this weighting scheme ensures that all the cycles spanning multiple components in the tree get non-zero circulation.

Nonzero Circulation of a cycle: First, we derive an upper bound on the circulation of any cycle completely contained in a subtree T of the working tree.

▶ **Lemma 11.** *The upper bound on the circulation of any cycle contained in a subtree T of the working tree $\mathcal{T}$ is $U_T = K^{h(r(T))} \times l(T)$.*

Proof. We prove this using induction on the height of $r(T)$.

Base case: The height of $r(T)$ is 1. Notice that this means that $r(T)$ has the maximum depth amongst all the nodes in $\mathcal{T}$, and therefore, $r(T)$ is a leaf node, and T is a singleton. Consider the two cases: i) when $r(T)$ is a planar graph, ii) when it is a constant sized graph.

By our weight assignment, if $r(T)$ is planar, the total weight of all the edges is zero. On the other hand, if $r(T)$ is a constant sized graph, the maximum circulation of a cycle is the sum of weights of its edges, that is, $\sum_{i=1}^{m}(K^0 \times 1 \times 2^i) < 2^{m+1} \leq K$. Thus, the circulation is upper bounded by $K^{h(r(T))} \times l(T)$ (as $l(T) = 1$).

Induction hypothesis: For any tree T' with $h(r(T')) \leq j-1$, the upper bound is $U_{T'} = K^{h(r(T'))} \times l(T')$.

Induction step: We will prove that for any tree T with $h(r(T)) = j$, the upper bound is $U_T = K^{h(r(T))} \times l(T)$.

Let the subtrees attached at $r(T)$ be $\{T_1, T_2, \ldots, T_k\}$. For any cycle in T, sum of the circulations of its projections on the subtrees $T_1, T_2, \ldots, T_k$ can be at most $\sum_{i=1}^{k} U_{T_i}$.

First, we handle the case when $r(T)$ is planar. For any subtree T_i, the total circulation of faces in $r(T)$ due to connection to T_i can be $6 \times K^{h(r(T_i))} \times l(T_i)$. This is because the circulation of each face adjacent to the separating set connecting with T_i is $2 \times K^{h(r(T_i))} \times l(T_i)$, and there can be at most 3 such faces. Thus,

$$
\begin{aligned}
U_T &= \sum_{i=1}^{k} U_{T_i} + \sum_{i=1}^{k}\left(6 \times K^{h(r(T_i))} \times l(T_i)\right) \\
&= \sum_{i=1}^{k}\left(K^{h(r(T_i))} \times l(T_i)\right) + \sum_{i=1}^{k}\left(6 \times K^{h(r(T_i))} \times l(T_i)\right) \\
&= 7 \times K^{h(r(T))-1} \times \sum_{i=1}^{k} l(T_i) \qquad (\because \forall i,\ h(r(T_i)) = h(r(T)) - 1) \\
&< K^{h(r(T))} \times \sum_{i=1}^{k} l(T_i) \qquad (\because K > 7) \\
&= K^{h(r(T))} \times l(T)
\end{aligned}
$$

Now, consider the case when $r(T)$ is a small non-planar graph. The maximum possible contribution from edges of $r(T)$ to the circulation of a cycle in T is less than $2^{m+1} \times K^{h(r(T))-1} \times l(T)$. Similar to the case when $r(T)$ is planar, contribution from all subtrees is at most $K^{h(r(T))-1} \times l(T)$. The total circulation of a cycle in T can be at most the sum of these two bounds, and is thus bounded above by $(2^{m+1}+1) \times K^{h(r(T))-1} \times l(T)$. Since, $K > 2^{m+2}$, the total possible circulation is less than $K^{h(r(T))} \times l(T)$.

Therefore, the upper bound $U_T = K^{h(r(T))} \times l(T)$. ◀

To see that each cycle gets a nonzero circulation, recall Lemma 8, which says that the circulation of the cycle is the sum of circulations of its projections on different components. Consider a cycle C. Recall that components with a non-empty projection of C form a subtree S_C in the component tree. From Claim 10, we can find a node $v^* \in S_C$ such that all other nodes in S_C are its descendants in the working tree $\mathcal{T}$. Thus, v^* is the unique minimum depth component on which C has a non-empty projection. Now, we show two things: (i) the contribution to the circulation from this component is nonzero, and (ii) it is larger than sum of all the circulation contributions from all its subtrees in the working tree.

Let v^* be the root of a subtree T in the working tree. Let the subtrees attached at $r(T)$ $(= v^*)$ be $\{T_1, T_2, \ldots, T_k\}$ and the separating sets in $r(T)$ at which they are attached be $\{\tau_1, \tau_2, \ldots, \tau_k\}$ respectively.

Case 1: when $r(T)$ is a constant-sized component. It is easy to see that the circulation of any cycle in this component will be nonzero as long as it takes a real edge, because the weights given are powers of 2. Also, the minimum weight of any edge in $r(T)$ is $2 \times \sum_{i=1}^{k} U_{T_i}$. Thus, when a cycle takes a real edge, contribution to its circulation from $r(T)$ is larger than the contribution from higher depth components (components in the subtrees attached at $r(T)$). And any cycle has to take a real edge, as the virtual edges and triangles all have disjoint set of vertices. (Here, the virtual triangle does not count as a cycle).

Case 2: when $r(T)$ is a planar component. The crucial observation here is that in a planar graph, all the faces inside a cycle contribute to its circulation in the same orientation.

▶ Claim 12 ([6]). In a planar graph, circulation of a cycle in clockwise orientation is the sum of circulations of the faces inside it (a proof can be found in [3]).

Since C passes through at least one of the subtrees attached at $r(T)$, say T_i, it must go through the separating set τ_i. Hence, the projection of C in $r(T)$, say C', must use the virtual edge (or one of the edges in the virtual triangle) corresponding to τ_i. This would imply that at least one of the faces adjacent to τ_i is inside C'. This is true for any subtree T_i which C passes through. As the faces adjacent to separating sets have nonzero circulations and each face has a positive circulation in clockwise direction, the circulation of C' is nonzero.

Recall that circulation of any face adjacent to τ_i is $2U_{T_i}$, where U_{T_i} is the upper bound on circulation contribution from T_i. This implies that the circulation of C' will surpass the total circulation from all the subtrees which C passes through. Thus, we can conclude the following.

▶ **Lemma 13.** *Circulation of any cycle which passes through at least two components is nonzero.*

Face circulations using edge weights: Now, we come back to the question of assigning weights to the edges in a planar component such that the faces get the desired circulations. Lemma 14 describes this procedure for any planar graph.

▶ **Lemma 14** ([19]). *Let $G(V, E)$ be a planar graph with F being its set of inner faces in some planar embedding. For any given function on the inner faces $w' : F \to \mathbb{Z}$, a skew-symmetric weight function $w \colon \vec{E} \to \mathbb{Z}$ can be constructed in log-space such that each face $f \in F$ has a circulation $w'(f)$ (a proof can be found in the full version [3]).*

This scheme can assign weight to any edge in the given graph, while we are not allowed to give weights to virtual edges/triangles. So, we first collapse all the virtual triangles to one node and all the virtual edges to one node. As no two virtual triangles/edges are adjacent, after this operation, every face remains a non-trivial face (except the virtual triangle face). Now, we apply the procedure from Lemma 14. After undoing the collapse, the circulations of the faces will not change and we will have the desired circulations.

Circulation of cycles contained within a single component: To construct w_1 for planar components, we assign +1 circulation to every face using Lemma 14 (similar to the case of multiple components). This would ensure nonzero circulation for every cycle within the planar component. This construction has been used in [19] for bipartite planar graphs. [29] also gives a log-space construction which ensures nonzero circulation for all cycles in a planar graph, using Green's theorem.

For the non-planar components, w_0 already ensures that each cycle has non-zero circulation. Therefore, we set $w_1 = 0$. Use a linear combination of w_0 and w_1 such that they do not interfere with each other. Such a combination is easy to achieve by multiplying w_1 by n^2 or a higher power of n since since w_0 is $O(n)$. This together with Lemma 13 gives us the following.

▶ **Lemma 15.** *Circulation of any cycle is non-zero.*

Complexity: The weights given by this scheme are polynomially bounded and the weight-construction procedure can be done in log-space (see the full version [3] for details).

3.3 Construction of the Working Tree

Now, we describe a log-space construction of the working tree. The idea is obtained from the construction of [21, Lemma 6], where they create a $O(\log n)$-depth tree of well-matched substrings of a given well-matched string. Recall that for a tree S, the working tree $\mathsf{wt}(S)$ is constructed by first choosing a center node $c(S)$ of S and marking it as the root of $\mathsf{wt}(S)$, and then recursively finding the working trees for each component obtained by removing the node $c(S)$ from S and connecting them to the root of $\mathsf{wt}(S)$, as subtrees.

First, consider the following possible definition of the center: for any tree S with n nodes, one can define its center to be a node whose removal would give disjoint components of size $\leq 1/2|S|$. Finding such a center is an easy task and can be done in log-space. Clearly, the depth of the working tree would be $O(\log n)$. It is not clear if the recursive procedure of finding centers for each resulting component can be done in log-space. Therefore, we give a more involved way of defining centers, so that the whole recursive procedure can be done in log-space.

First, we make the tree S rooted at an arbitrary node r. To find the child-parent relations of the rooted tree, one can do the standard log-space traversal of a tree.

Tree traversal [22]: for every node, give its edges an arbitrary cyclic ordering. Start traversing from the root r by taking an arbitrary edge. If you arrive at a node u using its edge e then leave node u using the right neighbor of e. This traversal ends at r with every edge being traversed exactly twice.

For any node v, let S_v denote the subtree of S, rooted at v. For any node v and one of its descendant nodes v' in S, let $S_{v,v'}$ denote the tree $S_v \setminus S_{v'}$. Moreover $S_{v,\epsilon}$ would just mean S_v, for any v. With our new definition of the center, at any stage of the recursive procedure, the component under consideration will always be of the form $S_{v,v'}$, for some nodes $v, v' \in S$. Now, we give a definition of the center for a rooted tree of the form $S_{v,v'}$.

Center $c(S_{v,v'})$: case (i) When $v' = \epsilon$, i.e. the given tree is S_v. Let c be a node in S_v, such that its removal gives components of size $\leq 1/2|S_v|$. If there are more than one such nodes then choose the lexicographically smallest one (there is at least one such center [17]). Define c as the center of $S_{v,v'}$.

Let the children of c in S_v be $\{c_1, c_2, \dots, c_k\}$. Clearly, after removing c from S_v, the components we get are $S_{c_1}, S_{c_2}, \dots, S_{c_k}$ and $S_{v,c}$. Thus, they are all of the form described above, and have size $\leq 1/2|S_v|$.

case (ii) When v' is an actual node in S_v. Let the node sequence on the path connecting v and v' be $(u_0, u_1, \dots, u_p)$, with $u_0 = v$ and $u_p = v'$. Let $0 \leq i < p$ be the least index such that $|S_{u_{i+1},v'}| \leq 1/2|S_{v,v'}|$. This index exists because $|S_{u_p,v'}| = 0$. Define u_i as the center of $S_{v,v'}$.

Let the children of u_i, apart from u_{i+1}, be $\{c_1, c_2, \dots, c_k\}$. After removal of u_i from $S_{v,v'}$, the components we get are $S_{c_1}, S_{c_2}, \dots, S_{c_k}$, $S_{u_{i+1},v'}$ and S_{v,u_i}. By the choice of i, $|S_{u_i,v'}| > 1/2|S_{v,v'}|$. Thus, $|S_{v,u_i}| \leq 1/2|S_{v,v'}|$. So, the only components for which we do not have a guarantee on their sizes, are $S_{c_1}, S_{c_2}, \dots, S_{c_k}$. Observe that when we find a center for the tree $S_{c_j,\epsilon}$ in the next recursive call, it will fall into case (i) and the components we get will have their sizes reduced by a factor of $1/2$.

Thus, we can conclude that in the recursive procedure for constructing the working tree, we reduce the size of the component by half in at most two recursive calls. Hence, the depth of working tree is $O(\log n)$. Now, we describe a log-space procedure for the working tree.

▶ **Lemma 16.** *For any tree S, its working tree* $\mathsf{wt}(S)$ *can be constructed in log-space.*

Proof. We just describe a log-space procedure for finding the parent of a given node x in the working tree. Running this procedure for every node will give us the working tree.

Find the center of the tree S. Removing the center would give many components. Find the component S_1, to which the node x belongs. Apply the same procedure recursively on S_1. Keep going to smaller components which contain x, till x becomes the center of some component. The center of the previous component in the recursion will be the parent of x in the working tree.

In this recursive procedure, to store the current component $S_{v,v'}$, we just need to store two nodes v and v'. Apart from these, we need to store center of the previous component and size of the current component.

To find the center of a given component $S_{v,v'}$, go over all possibilities of the center, depending on whether v' is ϵ or a node. For any candidate center c, find the sizes of the components generated if c is removed. Check if the sizes satisfy the specified requirements. Any of these components is also of the form $S_{u,u'}$ and thus can be stored with two nodes.

By the standard log-space traversal of a tree, for any given tree $S_{v,v'}$, one can count the number of nodes in it and test membership of a given node. Thus, the whole procedure works in log-space. ◀

4 Discussion

One of the open problems is to construct a polynomially bounded isolating weight assignment for a more general class of graphs, in particular, for all bipartite graphs. Our approach does not directly extend to more general minor-free graphs, because their decomposition can involve separating sets of size more than 3. For example, when we have a separating set of size 4, a cycle can have two different projections in a component, i.e., it enters the component twice and leaves the component twice. These two projections can contribute to the total circulation with opposite signs and can cancel each other.

The isolation question is also open for general planar graphs and small genus bipartite graphs.

5 Acknowledgements

We thank Arpita Korwar for various helpful discussions.

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